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On the Existence and Uniqueness of *M*-Splines

VIMALA ABRAHAM

Département de Mathématiques,
Centre-Universitaire, Sidi-Bel-Abbès, Algeria

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1. INTRODUCTION

In this paper we discuss the existence and uniqueness of the interpolating M-splines defined by Lucas [2]. Lucas has given two sets of sufficient conditions for the existence of M-splines. We obtain both necessary and sufficient conditions for the existence and uniqueness of such splines. Schaback [3] has generalized the concept of M-splines and studied the problem in the general setting of vector spaces. We restrict our attention to M-splines in Hilbert spaces.

Throughout this paper we will adhere to the following notations: If X is a real Hilbert space and M is a closed subspace of X, then $M^{\perp}$ denotes the orthogonal complement of M in X and P_M stands for the projection operator taking X onto M. For $x \in X$, the set $\Phi(x; M)$ is defined by

$$\Phi(x; M) = x + M.$$

If T is a continuous linear transformation of X to a second real Hilbert space Y, then $N(T)$ and $K(T)$ denote the kernel and cokernel of T, respectively. T_M stands for the restriction of T to M. T_M^{-1} is the continuous linear inverse of T_M, and for $A \subset Y$,

$$T_M^{-1}(A) = \{x \in M \mid T_M x \in A\}.$$

2. EXISTENCE AND UNIQUENESS

Let X be a real Hilbert space and let $M(x, y)$ be a continuous bilinear functional on $X \times X$. Suppose that A is a closed subspace of X such that

$$M(a, a) \geqslant 0, \qquad \forall a \in A. \tag{1.1}$$

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For each $x \in X$, $M(x, \cdot)$ is a continuous linear functional on X, and by the Riesz representation theorem there exists a continuous linear transformation τ from X to itself such that

$$M(x, y) = \langle \tau x, y \rangle, \qquad \forall x, y \in X. \tag{1.2}$$

Set

$$P_A \tau = T. \tag{1.3}$$

Then

$$M(x, a) = \langle Tx, a \rangle, \qquad \forall x \in X, \quad \forall a \in A, \tag{1.4}$$

and T is a continuous linear transformation from X to A. Set

$$\hat{A} = \{\hat{a} \in A \mid M(\hat{a}, a) = 0, \forall a \in A\}, \tag{1.5}$$

$$\tilde{A} = \{\tilde{a} \in A \mid M(a, \tilde{a}) = 0, \forall a \in A\}, \tag{1.6}$$

and

$$A_1 = \{a_1 \in A \mid M(a_1, a_1) = 0\}. \tag{1.7}$$

We have trivially,

$$\hat{A} \subset A_1 \qquad \text{and} \qquad \tilde{A} \subset A_1. \tag{1.8}$$

We first state a few lemmas which will be used in the course of this paper.

LEMMA 1.1.

$$\hat{A} = A \cap N(T) = \tilde{A} = (TA)^{\perp} \cap A. \tag{1.9}$$

Proof. The relations

$$\hat{A} = A \cap N(T) \qquad \text{and} \qquad \tilde{A} = (TA)^{\perp} \cap A$$

are obvious. It remains to be shown that $\hat{A} = \tilde{A}$.

It is sufficient to prove that

$$M(a, \hat{a}) = 0, \qquad \forall a \in A, \quad \forall \hat{a} \in \hat{A},$$

and

$$M(\tilde{a}, a) = 0, \qquad \forall \tilde{a} \in \tilde{A}, \quad \forall a \in A.$$

Let $\hat{a} \in \hat{A}$ and $a \in A$. Then for any real α,

$$\begin{aligned} 0 &\leqslant M(\alpha\hat{a} + a, \alpha\hat{a} + a) \\ &= \alpha^2 M(\hat{a}, \hat{a}) + \alpha M(\hat{a}, a) + \alpha M(a, \hat{a}) + M(a, a) \\ &= \alpha M(a, \hat{a}) + M(a, a). \end{aligned} \tag{1.10}$$

Since (1.10) holds for any real α, we must have

$$M(a, \hat{a}) = 0, \qquad \forall a \in A, \quad \forall \hat{a} \in \hat{A}.$$

Similarly, we can show that $\tilde{A} \subset \hat{A}$. Hence the result.

LEMMA 1.2. *The following three conditions are equivalent:*

(1) $$M(a, a_1) = 0, \qquad \forall a \in A, \quad \forall a_1 \in A_1, \tag{1.11}$$

(2) $$\hat{A} = A_1, \tag{1.12}$$

(3) $$M(a_1, a) = 0, \qquad \forall a_1 \in A_1, \qquad \forall a \in A. \tag{1.13}$$

LEMMA 1.3. *If $M(x, y)$ is symmetric, then $\hat{A} = A_1$.*

The M-spline of Lucas is defined as follows:

DEFINITION 1.4. An element $s \in X$ is called an *M-spline* if

$$M(s, a) = 0, \qquad \forall a \in A. \tag{1.14}$$

Denote by S the set of all M-splines in X. Then

$$S = N(T) \tag{1.15}$$

and S is a closed subspace of X.

Statement of the problem. The M-spline interpolation problem is to find for each $a^\perp \in A^\perp$ an element $s \in \Phi(a^\perp; A)$ such that $M(s, a) = 0$, $\forall a \in A$. It is clear that the problem of finding an M-spline in $\Phi(a^\perp; A)$ for each $a^\perp \in A^\perp$ is equivalent to the problem of finding an M-spline in $\Phi(x; A)$ for each $x \in X$.

The following theorem gives necessary and sufficient conditions in terms of the continuous linear transformation T for the existence and uniqueness of interpolating M-splines. It also gives the representation for the M-splines in $\Phi(a^\perp; A)$.

THEOREM 1.5. *For each $a^\perp \in A^\perp$ there exists an M-spline in $\Phi(a^\perp; A)$ if and only if*

$$T(A^\perp) \subset TA. \tag{1.16}$$

The set of M-splines belonging to $\Phi(a^\perp, A)$ is given by

$$S_{a^\perp} = a^\perp - T_A^{-1}(Ta^\perp). \tag{1.17}$$

A necessary and sufficient condition for $S_{a^\perp}$ to reduce to a singleton is

$$\hat{A} = \{0\}. \tag{1.18}$$

If TA is closed and (1.18) *holds, then there exists an unique M-spline*

$$s_{a^\perp} = a^\perp - T_A^{-1}(Ta^\perp) \tag{1.19}$$

in $\Phi(a^\perp; A)$ which depends continuously on $a^\perp$.

Proof. Assume first that $T(A^\perp) \subset TA$. Then for each $a^\perp \in A^\perp$ there exists $a \in A$ such that $Ta^\perp = Ta$. Set $s = a^\perp - a$. Then $s \in \Phi(a^\perp; A)$ and $s \in N(T) = S$. In other words, s is an M-spline in $\Phi(a^\perp; A)$. Conversely, if there exists a solution to the M-spline interpolation problem, then given $a^\perp \in A^\perp$, there exists $s \in \Phi(a^\perp; A)$ such that $Ts = 0$, i.e., there exists $a \in A$ such that $s = a^\perp + a$ with $Ta^\perp + Ta = 0$. This implies that $T(A^\perp) \subset TA$. Now consider the set $S_{a^\perp}$ defined by (1.17). If $s \in S_{a^\perp}$, then $s \in \Phi(a^\perp; A)$ and $Ts = 0$. Hence $S_{a^\perp}$ consists of M-splines in $\Phi(a^\perp; A)$. On the other hand, if s is an M-spline in $\Phi(a^\perp; A)$, then $s = a^\perp + a_0$ with $Ts = Ta^\perp + Ta_0 = 0$. Hence $-a_0 \in \{a \in A \mid T_A a = Ta^\perp\} = T_A^{-1}(Ta^\perp)$. Thus the set of M-splines in $\Phi(a^\perp; A)$ is precisely $S_{a^\perp}$. We have $S_{a^\perp} = a^\perp - a' + N(T_A) = a^\perp - a' + \hat{A}$ where a' is the unique element of $K(T_A)$ such that $T_A a' = Ta^\perp$. Hence $S_{a^\perp}$ reduces to a singleton if and only if $\hat{A} = \{0\}$.

Finally, $\hat{A} = \{0\}$ implies $\tilde{A} = \{0\}$ (Lemma 1.1). If further, TA is closed, then $TA = A$ and by the open mapping theorem, T_A^{-1} exists and is continuous. Hence the unique spline $s_{a^\perp} \in \Phi(a^\perp; A)$ is defined by $s_{a^\perp} = a^\perp - T_A^{-1}(Ta^\perp)$ and depends continuously on $a^\perp$.

THEOREM 1.6. *A necessary condition for the M-spline interpolation problem to have a solution is*

$$M(x, \hat{a}) = 0, \qquad \forall x \in X, \quad \forall \hat{a} \in \hat{A}. \tag{1.20}$$

If TA is closed, then the condition (1.20) *is also sufficient.*

Proof. If there exists an M-spline in $\Phi(a^\perp; A)$ for each $a^\perp \in A^\perp$, then by Theorem 1.5,

$$T(A^\perp) \subset TA.$$

Hence

$$\langle Ta^{\perp}, b\rangle = 0, \qquad \forall a^{\perp}\in A^{\perp}, \quad \forall b\in (TA)^{\perp}\cap A,$$

and by Lemma 1.1,

$$\langle Ta^{\perp}, \hat{a}\rangle = 0, \qquad \forall a^{\perp}\in A^{\perp}, \quad \forall \hat{a}\in \hat{A}.$$

Again by Lemma 1.1, $\langle Ta, \hat{a}\rangle = 0$, $\forall a\in A$, $\forall \hat{a}\in\hat{A}$. Since each $x\in X$ has the unique representation

$$x = a_x^{\perp} + a_x, \qquad a_x^{\perp}\in A^{\perp}, \quad a_x\in A,$$

we have

$$\langle Tx, \hat{a}\rangle = 0, \qquad \forall x\in X, \quad \forall \hat{a}\in\hat{A}.$$

To prove the second part we have for $a^{\perp}\in A^{\perp}$, $Ta^{\perp} = Ta + b$, $a\in A$, $b\in (TA)^{\perp}$ since TA is closed. Then $b\in\hat{A}$ and by (1.20) and Lemma 1.1, the identity

$$\langle Ta^{\perp}, b\rangle = 0 = M(a, b) + \langle b, b\rangle = \langle b, b\rangle$$

implies $b = 0$. Hence $T(A^{\perp})\subset TA$ and the result follows from Theorem 1.5.

We observe that each of the two sets of conditions used by Lucas [2] to prove the existence of M-splines imply the condition (1.20). Theorem 1.6 motivates us to look for conditions on $M(x, y)$ which will ensure that TA is closed. Any one of the following five sets of conditions imply the existence of a set $S_{a^{\perp}}$ of M-splines in $\Phi(a^{\perp}; A)$.

HYPOTHESIS A. *The condition* (1.20) *holds. There exists a closed subspace* $\bar{A}$ *of* A *such that*

$$A = \hat{A}\oplus\bar{A}, \tag{1.21}$$

and there exists $K > 0$ *such that*

$$M(a, a)\geqslant K\,\|a\|^2, \qquad \forall a\in\bar{A}. \tag{1.22}$$

HYPOTHESIS B. *The condition* (1.20) *holds. There exists a closed subspace* $\bar{A}$ *of* A *such that*

$$A = \hat{A}\oplus\bar{A},$$

and there exists $a_0\in A$ *and* $K > 0$ *such that*

$$|M(a, a_0)|\geqslant K\,\|a\|, \quad \forall a\in\bar{A}. \tag{1.23}$$

HYPOTHESIS C. *$M(x,y)$ satisfies* (1.20) *and any one of the following three conditions:*

$$\text{(i)} \quad M(x,y) \text{ is symmetric,} \tag{1.24}$$

$$\text{(ii)} \quad M(a, a_1) = 0, \qquad \forall a \in A, \quad \forall a_1 \in A_1, \tag{1.25}$$

$$\text{(iii)} \quad M(a_1, a) = 0, \qquad \forall a_1 \in A_1, \quad \forall a \in A. \tag{1.26}$$

There exists a closed subspace A_2 of A such that

$$A = A_1 \oplus A_2, \tag{1.27}$$

and there exists $K > 0$ such that

$$M(a,a) \geqslant K \|a\|^2, \qquad \forall a \in A_2. \tag{1.28}$$

HYPOTHESIS D. *$M(x,y)$ satisfies* (1.20) *and any one of the three conditions* (1.24), (1.25), (1.26). *There exists a closed subspace A_2 of A such that*

$$A = A_1 \oplus A_2,$$

and there exists $a_0 \in A$ and $K > 0$ such that

$$|M(a, a_0)| \geqslant K \|a\|, \qquad \forall a \in A_2. \tag{1.29}$$

HYPOTHESIS E. *$M(x,y)$ satisfies* (1.24), (1.27), *and*

$$M(x,x) \geqslant 0, \qquad \forall x \in X. \tag{1.30}$$

In addition, one of the two conditions (1.28), (1.29) *holds.*

The Hypothesis C when $M(x,y)$ satisfies (1.20), (1.25), (1.27), and (1.28) is used by Lucas [2] to prove his main result on the existence of a set of M-splines in $\Phi(a^\perp; A)$. Also, the Hypothesis E with $M(x,y)$ satisfying (1.24), (1.27), (1.28), (1.30) is due to him [2].

If the interpolating M-spline is to be unique, we need the additional condition $\hat{A} = \{0\}$. Conditions which imply the existence of a unique interpolating M-spline are:

HYPOTHESIS F. *There exists $K > 0$ such that*

$$M(a,a) \geqslant K \|a\|^2, \qquad \forall a \in A. \tag{1.31}$$

HYPOTHESIS G. *There exists $a_0 \in A$ and $K > 0$ such that*

$$|M(a, a_0)| \geqslant K \|a\|, \qquad \forall a \in A. \tag{1.32}$$

Hypothesis F is due to Lucas [2]. That the hypotheses given above imply the existence of M-splines follows from the following facts:

(1) Each one of the formulas (1.22), (1.23), (1.28), (1.29), (1.31), (1.32) implies $\|Ta\| \geqslant K\,\|a\|$ for some closed subspace $A_0 \ni a$ of A. Then T is invertible on TA_0 and TA_0 is closed.

(2) The special choice of A_0 in the various cases always implies that $TA_0 = TA$.

EXAMPLE. It is known (see [2]) that the interpolating splines of Attéia are examples of M-splines. By going over to product spaces the smoothing splines of Attéia can also be written as M-splines. Then the results of this paper can be applied to give conditions for the existence of smoothing splines, too. The details are left to the reader.

We remark finally that an external result similar to that of Lucas can also be obtained.

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